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Lagrangian Description of World-Line Deviations

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Abstract We introduce a Lagrangian which can be varied to give both the equation of motion and world-line deviations of spinning particles simultaneously.

Keywords World-line deviation · Spinning particles · Combined actions

1 Introduction

The equation of geodesic deviation is a well-known equation of general relativity with important applications, namely, it describes the relative motion of many particles. This equation could be derived by a variety of methods including the second covariant variation of the point particle Lagrangian. It could also be derived by first variation of a combined action that gives both the equation of motion and the deviation equation at once [1]. The Lagrangian approach has several advantages, namely it allows the deviation to be formulated for other dynamical systems with more general objects [2].

In a recent publication [3] the geodesic deviation equation was generalized to a deviation equation for the world-lines of spinning particles. This was basically achieved by considering a one-parameter family of world-lines of spinning particles and demanding the invariance of the equations of motion, the Mathisson–Papapetrou–Dixon equations [4], on different values of the parameter. The same technique was applied in [5] to the Dixon–Souriau equations [6] to obtain the world-line deviations of charged spinning particles.

In the present work we introduce a combined Lagrangian and show that both the equation of motion and the deviation equation of spinning particles can be derived from this Lagrangian. In this Lagrangian approach the spin is represented by the gyration of an orthonormal tetrad attached to the particle world-lines.

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2 The Action

We begin with a matter Lagrangian of the form

$$\mathcal{L} = n\sqrt{-g}L(e^a_\mu, \dot{e}^a_\mu, u^\mu), \tag{1}$$

where n is the particle number density, u^μ are the particles four-velocity satisfying

$$u_\mu u^\mu = -1, \tag{2}$$

$\nabla_\alpha e^a_\mu$ are the covariant derivatives and

$$\dot{e}^a_\mu = u^\alpha \nabla_\alpha e^a_\mu. \tag{3}$$

We adopt an orthonormal tetrad field given by

$$\begin{aligned} e^a_\mu e_{a\nu} &= g_{\mu\nu}, \\ e_{a\mu} e^\mu_b &= \eta_{ab}. \end{aligned} \tag{4}$$

The action

$$S = \int \mathcal{L} d^4x = \int nL(e^a_\mu, \dot{e}^a_\mu, u^\mu)\sqrt{-g}d^4x \tag{5}$$

describes the dynamics of a dust of particles with spin in a fixed background space-time. We need not to specify the explicit form of the Lagrangian here. Taking the relation (2) into account, we impose the following constraint on the Lagrangian [7]

$$u^\mu \frac{\partial L}{\partial u^\mu} = 0. \tag{6}$$

The momentum four-vector and the spin tensor are defined by

$$P^\mu = \frac{\partial L}{\partial u_\mu} - Lu^\mu, \tag{7}$$

$$S^{\mu\nu} = \frac{\partial L}{\partial \dot{e}^\nu_a} e^\mu_a - \frac{\partial L}{\partial \dot{e}^\mu_a} e^\nu_a, \tag{8}$$

respectively. Translational equations of motion may be obtained by extremizing the action on variation of world-lines [7, 8]. For a one-parameter family of tetrad fields $e^a_\alpha(x, \lambda)$ and congruences $x^\mu(t, \lambda)$ we hold the tetrad attached to world-lines fixed by parallel propagation. Under infinitesimal variation

$$x^\mu(t, \lambda) = x^\mu(t, 0) + \lambda n^\mu(t), \tag{9}$$

various quantities vary as follows

$$\delta(e^a_\alpha) = 0, \tag{10}$$

$$\delta(d\tau) = -\lambda u^\alpha \nabla_\alpha n_\mu u^\mu d\tau, \tag{11}$$

$$\delta(u^\mu) = \lambda h^\mu_\alpha u^\nu \nabla_\nu n^\alpha, \tag{12}$$

$$\delta(\dot{e}^a_\mu) = \lambda(R^\nu{}_{\mu\alpha\beta} u^\alpha n^\beta e^a_\nu + \dot{e}^a_\mu u_\nu u^\beta \nabla_\beta n^\nu), \tag{13}$$

$$\delta(n\sqrt{-g}d^4x) = -\lambda u^\mu u_\alpha \nabla_\mu n^\alpha n\sqrt{-g}d^4x \tag{14}$$

in which $h^\mu_\nu = \delta^\mu_\nu + u_\nu u^\mu$ is a projection. The last equation above follows from the conservation of the number of particles in an infinitesimal flux tube. Equation (13) may be derived as follows

$$\begin{aligned}
\delta(\dot{e}^a_\alpha) &= \delta(u^\mu \nabla_\mu e^a_\alpha) \\
&= \delta(u^\mu)\nabla_\mu e^a_\alpha + u^\mu \delta(\nabla_\mu e^a_\alpha) \\
&= \lambda(\delta^\mu_\alpha + u_\alpha u^\mu) u^\nu \nabla_\nu n^\alpha \nabla_\mu e^a_\alpha + \lambda u^\mu n^\kappa \nabla_\kappa \nabla_\mu e^a_\alpha \\
&= \lambda(u^\kappa \nabla_\kappa n^\mu \nabla_\mu e^a_\alpha + u^\mu n^\kappa \nabla_\kappa \nabla_\mu e^a_\alpha + u^\kappa \nabla_\kappa n^\beta u_\beta u^\mu \nabla_\mu e^a_\alpha) \\
&= \lambda(-u^\kappa n^\mu \nabla_\kappa \nabla_\mu e^\mu_\alpha + u^\mu n^\kappa \nabla_\kappa \nabla_\mu e^\mu_\alpha) + \lambda \dot{e}^a_\alpha u^\beta \nabla_\beta n_\mu u^\mu + (\text{div.}) \\
&= \lambda R^\beta{}_{\alpha\mu\nu} u^\mu n^\nu e^a_\beta + \lambda \dot{e}^a_\alpha u^\beta \nabla_\beta n_\mu u^\mu.
\end{aligned}$$

With the aid of these relations we are able to calculate the variation of the action. The result is

$$\delta S = \lambda \int n\left(\frac{\partial L}{\partial u^\mu} u^\alpha \nabla_\alpha n^\mu + \frac{1}{2}\left(\frac{\partial L}{\partial \dot{e}^a_{[\mu}} e^a_{\gamma]}\right) R^\gamma{}_{\mu\nu\alpha} u^\nu n^\alpha\right)\sqrt{-g}\, d^4x \tag{15}$$

in which use has been made of relation (6). In terms of momentum and spin tensor, this may be rewritten as

$$\delta S = \lambda \int n\left(P_\mu \dot{n}^\mu - \frac{1}{2} S^{\alpha\beta} R^\mu{}_{\nu\alpha\beta} u^\nu n^\mu\right)\sqrt{-g}\, d^4x \tag{16}$$

or

$$\delta S = -\lambda \int n n_\mu \left(\dot{P}^\mu + \frac{1}{2} S^{\alpha\beta} R^\mu{}_{\nu\alpha\beta} u^\nu\right)\sqrt{-g}\, d^4x + (\text{div.}). \tag{17}$$

Inspired by this relation, and noting the relation $n\sqrt{-g}\, d^4x = N d\tau$ in which N is the conserved number of particles in an infinitesimal flux tube, we consider the following combined Lagrangian

$$S_c = -\int n_\mu \left(\dot{P}^\mu + \frac{1}{2} S^{\alpha\beta} R^\mu{}_{\nu\alpha\beta} u^\nu\right) d\tau. \tag{18}$$

This is consistent with the general form of the relative motion Lagrangian given in [9].

3 Equations of Motion and Deviation

The translational equation of motion is obtained from (18) by variation with respect to n^μ. This result in

$$\dot{P}^\mu = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu S^{\alpha\beta}, \tag{19}$$

the MPD equation of motion for the momentum. A variation of world-line, of the form of (9) however, results in

$$\frac{D}{D\lambda}\left(\dot{P}^\mu + \frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu S^{\alpha\beta}\right) = 0. \tag{20}$$

On the other hand, it can be shown that

$$\frac{D}{D\lambda}\frac{D}{D\tau}P^{\mu} = \frac{D}{D\tau}\frac{D}{D\lambda}P^{\mu} + R^{\mu}{}_{\nu\alpha\beta}u^{\nu}n^{\alpha}P^{\beta}, \tag{21}$$

where $\frac{D}{D\tau} = u^{\alpha}\nabla_{\alpha}$, $\frac{D}{D\lambda} = n^{\alpha}\nabla_{\alpha}$. By inserting this into the previous equation and using the abbreviation $j^{\mu} = \frac{D}{D\lambda}P^{\mu}$ we obtain

$$\frac{Dj^{\mu}}{D\tau} = R^{\mu}{}_{\nu\alpha\beta}u^{\nu}n^{\alpha}P^{\beta} + n^{\kappa}\nabla_{\kappa}\left(\frac{1}{2}R^{\mu}{}_{\nu\alpha\beta}u^{\nu}S^{\alpha\beta}\right) \tag{22}$$

which is the deviation equation, in agreement with the result of Ref. [3]. It can be shown that the spin equation of motion

$$\dot{S}^{\mu\nu} = P^{\mu}u^{\nu} - P^{\nu}u^{\mu} \tag{23}$$

follows from the fact that the energy-momentum tensor derived from the action (5) should be symmetric [8]. One may incorporate this into the above Lagrangian as a constraint to obtain the relevant deviation equation.

4 Discussion

We started with a Lagrangian whose specific form was not given. We then showed that its variation with respect to the particle world-lines lead to a new combined Lagrangian from which both the equation of motion and the deviation equation can be recovered simultaneously. This approach benefits from the usual advantages of Lagrangian formulations, say, it is more powerful for studying the symmetries of the system. It also allows the deviation equation to be obtained for more general dynamical systems.

The above combined Lagrangian is re-parametrization invariant. This allows to switch from the normalized four-velocity to a non-normalized one which is more convenient when we deal with spinning particles.

Most of the above procedure could be easily generalized to the world-line deviations of a Weyssenhoff-type spinning fluid by starting with a more general Lagrangian which is not necessarily homogeneous in particle number density, and to charged spinning particles.

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